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Split Rank One Semisimple Symmetric Spaces and c-Functions

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ABSTRACT: *Split rank one semisimple symmetric spaces and their structures are discussed and a method of computing c-functions for such symmetric spaces are explained. Most parts of this note are based on a joint work with T. Oshima.*

§1. Split rank one semisimple symmetric spaces. Let $\mathfrak{g}$ be a semisimple Lie algebra and let σ be its involution. Denote by $\mathfrak{h}$ and $\mathfrak{q}$ the (+1)- and (-1)-eigenspaces of σ, respectively. Then $(\mathfrak{g},\mathfrak{h})$ is called a *(semisimple) symmetric pair*. It is known the existence of a Cartan involution θ of $\mathfrak{g}$ commuting with σ. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition. Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}\cap\mathfrak{q}$ and a maximal abelian subspace $\mathfrak{j}$ of $\mathfrak{q}$ containing $\mathfrak{a}$. Then *dim* $\mathfrak{j}$ (resp. *dim* $\mathfrak{a}$) is called the *rank* (resp. *split rank*) of $(\mathfrak{g},\mathfrak{h})$. In particular, $(\mathfrak{g},\mathfrak{h})$ is of *split rank one* if *dim* $\mathfrak{a}=1$. There are a lot of such symmetric pairs which we are going to list up (cf. [OS2, Table II]): $I_1(p,q)=(\mathfrak{so}(p+1,q+1),\mathfrak{so}(p+1,q))$, $I_1^d(p,q)=(\mathfrak{so}(p+q+1,1),\mathfrak{so}(p+1)\oplus\mathfrak{so}(q,1))$, $I_2(p,q)=(\mathfrak{su}(p+1,q+1),\mathfrak{u}(p+1,q))$, $I_1^d(p,q)=(\mathfrak{su}(p+q+1,1),\mathfrak{su}(p+1)\oplus\mathfrak{su}(q,1)\oplus i\mathbb{R})$, $I_3(p,q)=(\mathfrak{sp}(p+1,q+1),\mathfrak{sp}(p+1,q)\oplus\mathfrak{sp}(1))$, $I_3^d(p,q)=(\mathfrak{sp}(p+q+1,1),\mathfrak{sp}(p+1)\oplus\mathfrak{sp}(q,1))$, $I_4^1=(\mathfrak{f}_{4(-20)},\mathfrak{so}(9))$, $I_4^2=(\mathfrak{f}_{4(-20)},\mathfrak{so}(8,1))$, $II_1(p)=(\mathfrak{sl}(p+2,\mathbb{R}),\mathfrak{gl}(p+1,\mathbb{R}))$, $II_1^d(p)=(\mathfrak{su}(p+1,1),\mathfrak{so}(p+1,1))$, $II_2(p)=(\mathfrak{sp}(p+2,\mathbb{R}),\mathfrak{sp}(p+1,\mathbb{R})\oplus\mathfrak{sp}(1,\mathbb{R}))$, $II_2^d(p)=(\mathfrak{sp}(p+1,1),\mathfrak{u}(p+1,1))$, $II_3=(\mathfrak{f}_{4(4)},\mathfrak{so}(5,4))$, $II_3^d=(\mathfrak{f}_{4(-20)},\mathfrak{sp}(2,1)\oplus\mathfrak{sp}(1))$, $III_1(p)=(\mathfrak{so}(p+2,\mathbb{C}),\mathfrak{so}(p+1,\mathbb{C}))$, $III_1^d(p)=$

$(\mathfrak{so}(p+1,1)\oplus\mathfrak{so}(p+1,1),\mathfrak{so}(p+1,1))$, $III_2(p)=(\mathfrak{sl}(p+2,\mathbb{C}),\mathfrak{gl}(p+1,\mathbb{C}))$, $III_2^d(p)=(\mathfrak{su}(p+1,1)\oplus\mathfrak{su}(p+1,1),\mathfrak{su}(p+1,1))$, $III_3(p)=(\mathfrak{sp}(p+2,\mathbb{C}),$ $\mathfrak{sp}(p+1,\mathbb{C})\oplus\mathfrak{sp}(1,\mathbb{C}))$, $III_3^d(p)=(\mathfrak{sp}(p+1,1)\oplus\mathfrak{sp}(p+1,1),\mathfrak{sp}(p+1,1))$, $III_4=$ $(\mathfrak{f}_4,\mathfrak{so}(9,\mathbb{C}))$, $III_4^d=(\mathfrak{f}_{4(-20)}\oplus\mathfrak{f}_{4(-20)},\mathfrak{f}_{4(-20)})$, $IV_1(p)=(\mathfrak{so}^*(2p+4),$ $\mathfrak{so}^*(2p+2)\oplus\mathfrak{so}^*(2))$, $IV_1^d(p)=(\mathfrak{so}(2p+2,2),\mathfrak{u}(p+1,1))$, $IV_2(p)=(\mathfrak{su}^*(2p+4),$ $\mathfrak{su}^*(2p+2)\oplus\mathfrak{su}^*(2)\oplus\mathbb{R})$, $IV_2^d(p)=(\mathfrak{su}(2p+2,2),\mathfrak{sp}(p+1,1))$, $IV_3=(\mathfrak{e}_{6(-26)},$ $\mathfrak{so}(9,1)\oplus\mathbb{R})$, $IV_3^d=(\mathfrak{e}_{6(-14)},\mathfrak{f}_{4(-20)})$, $V_1=(\mathfrak{sl}(3,\mathbb{C}),\mathfrak{sl}(3,\mathbb{R}))$, $V_2=(\mathfrak{su}(3,3),$ $\mathfrak{sp}(3,\mathbb{R}))$, $V_2^d=(\mathfrak{su}^*(6),\mathfrak{sl}(3,\mathbb{C})\oplus i\mathbb{R})$, $V_3=(\mathfrak{e}_{6(2)},\mathfrak{f}_{4(4)})$, $V_3^d=(\mathfrak{e}_{6(-26)},$ $\mathfrak{su}^*(6)\oplus\mathfrak{su}(2))$.

In this note, it is assumed that $(\mathfrak{g},\mathfrak{h})$ *is of split rank one unless otherwise stated*. Let G be a connected Lie group with $Lie G=\mathfrak{g}$. Suppose that σ is lifted to G and write its lifting by the same letter. Then G/G^σ ($G^\sigma=\{g\in G;\sigma(g)=g\}$) is called a *(semisimple) symmetric space* belonging to $(\mathfrak{g},\mathfrak{h})$. In general, for a given $(\mathfrak{g},\mathfrak{h})$, there are non-isomorphic symmetric spaces belonging to $(\mathfrak{g},\mathfrak{h})$. In the case where $G=\mathcal{I}nt\,\mathfrak{g}$, $F(\mathfrak{g},\mathfrak{h})=\pi_1(G/G^\sigma)$ is generated by one element and depends only on $(\mathfrak{g},\mathfrak{h})$. There is a one to one correspondence between the totality of non isomorphic symmetric spaces belonging to $(\mathfrak{g},\mathfrak{h})$ and that of subgroups of $F(\mathfrak{g},\mathfrak{h})$. For this reason, a classification of symmetric spaces of split rank one is accomplished if $F(\mathfrak{g},\mathfrak{h})$ is determined for each pair $(\mathfrak{g},\mathfrak{h})$. Now the result is as follows (cf. [Se3]): <u>(A.1) $F(\mathfrak{g},\mathfrak{h})=\mathbb{Z}$</u> if $(\mathfrak{g},\mathfrak{h})$ is one of $I_1(p,1)$, $II_1^d(p)$, $III_2^d(p)$, $IV_2^d(p)$, IV_3^d. <u>(A.2) $F(\mathfrak{g},\mathfrak{h})=\mathbb{Z}_3$</u> if $(\mathfrak{g},\mathfrak{h})$ is one of V_i $(i=1,2,3)$. <u>(A.3) $F(\mathfrak{g},\mathfrak{h})=\mathbb{Z}_2$</u> if $(\mathfrak{g},\mathfrak{h})$ is one of $I_1(p,q)$ $(q>1)$, $II_1(p)$ $(p>1)$, $II_2^d(p)$, $III_1(p)$, $III_1^d(p)$ $(p>1)$, $III_3^d(p)$, V_2^d. <u>(A.4) $F(\mathfrak{g},\mathfrak{h})=1$</u> otherwise. For example, consider III_i^d $(i = 1, 2, 3, 4)$. In these cases, symmetric spaces are nothing but group manifolds. Let G be a connected simple Lie group whose Lie algebra $\mathfrak{g}$ is of *split rank one*. Then $\mathfrak{g}$ is isomorphic to one of $\mathfrak{so}(p,1)$, $\mathfrak{su}(p,1)$, $\mathfrak{sp}(p,1)$, $\mathfrak{f}_{4(-20)}$. Since $G\times G/\Delta G\simeq G$, a group manifold G with actions of $G\times G$ from both sides is a special case of semisimple symmetric spaces. If $\mathfrak{g}=\mathfrak{so}(p,1)$ $(p>2)$, then G is $SO_0(p,1)$ or $Spin_0(p,1)$, the latter is simply connected. If $\mathfrak{g}=\mathfrak{sp}(p,1)$ $(p>1)$ or $\mathfrak{f}_{4(-20)}$, then $G=\mathcal{I}nt\,\mathfrak{g}$. In the case where $\mathfrak{g}=\mathfrak{su}(p,1)$, there are infinitely many non-isomorphic choices of G because $\mathcal{I}nt\,\mathfrak{g} = SU(p,1)/\mathbb{Z}_{p+1}$, $\pi_1(\mathcal{I}nt\,\mathfrak{g})=\mathbb{Z}$. Return to our situation. It is easy to show (B.1) if $\#F(\mathfrak{g},\mathfrak{h})<\infty$ and if G is a real form of a simply connected complex Lie group, then G/G_0^{σ} is simply connected and (B.2) if $\#F(\mathfrak{g},\mathfrak{h})=\infty$, then $\mathfrak{l}$ is not semisimple and the center of the motion group of the universal covering of $(\mathcal{I}nt\,\mathfrak{g})/(\mathcal{I}nt\,\mathfrak{g})^{\sigma}$ has infinite center.

From the assumption, there is $Y\in\mathfrak{a}$ such that {eigenvalues of $ad_{\mathfrak{g}}(Y)$} is $\{0,\pm1\}$ or $\{0,\pm1,\pm2\}$ and that $\mathfrak{a}=\mathbb{R}Y$. Put $\mathfrak{g}_j=\{Z\in\mathfrak{g};[Y,Z]=jZ\}$ $(j=0,\ \pm1,\ \pm2)$. Then $\sigma\theta$ leaves each $\mathfrak{g}_j$ invariant. Since $(\sigma\theta)^2=1$, define $\mathfrak{g}_j^{\pm}=\{Z\in\mathfrak{g}_j;\sigma\theta Z=\pm Z\}$ and put $m_j^{\pm}=\dim\mathfrak{g}_j^{\pm}$, $m_j=m_j^{+}+m_j^{-}$ $(j=1,\ 2)$. Then the following hold: (C.1) $m_1^{+}+m_1^{-}>0$. (C.2) If $m_2^{-}>0$, then $m_1^{+}=m_1^{-}$. (C.3) $m_1^{-}=m_2^{-}=0$ if and only if $(\mathfrak{g},\mathfrak{h})$ is Riemannian. (C.4) If $m_1^{+}=m_2^{-}=0$, then $(\mathfrak{g},\mathfrak{h})$ is of $\mathfrak{l}_{\varepsilon}$-type (for the definition of $\mathfrak{l}_{\varepsilon}$-type, see [OS1]). Put $\mathfrak{n}_{\sigma}=\mathfrak{g}_1\oplus\mathfrak{g}_2$, $\bar{\mathfrak{n}}_{\sigma}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_{-2}$ and denote by A, N_{σ}, $\bar{N}_{\sigma}$ the analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{n}_{\sigma}$, $\bar{\mathfrak{n}}_{\sigma}$, respectively. Let P_{σ} be the parabolic subgroup of G whose unipotent radical is N_{σ} and let $P_{\sigma}=M_{\sigma}A_{\sigma}N_{\sigma}$ be its Langlands decomposition. May assume that there is a closed subgroup A_1 of G^{σ} such that $A_{\sigma}=A_1A$. Moreover, (D) if $(\mathfrak{g},\mathfrak{h})$ is split rank=1 but rank>1, then M_{σ} is connected. I am now going to mention connected components of M_{σ} for the remaining cases. For simplicity, we assume that *G/H* ($H=G_0^\sigma$) *is simply connected and of rank one*. Then $M_\sigma \cap H = M_{\sigma,0}$ and $\#(M_\sigma / M_\sigma \cap H)$ coincides with the number of open H-orbits of $G/P_{\sigma,0}$. In particular, (E.1) for simply connected symmetric spaces $SO_0(p+1,q+1)/SO_0(p+1,q)$ $(q>1)$, $Sp(p+2,\mathbb{R})/Sp(p+1,\mathbb{R})\times Sp(1,\mathbb{R})$, $F_{4(4)}/Spin_0(5,4)$, there are two open H-orbits of $G/P_{\sigma,0}$, (E.2) for simply connected symmetric spaces $SL(p+2,\mathbb{R})/GL_+(p+1,\mathbb{R})$ $(p>1)$, there are four open H-orbits and (E.3) for a simply connected symmetric space $SO_0(p+1,2)/SO_0(p+1,1)$, M_σ has infinitely many connected components.

§2. **c-functions for Riemannian symmetric spaces.** Consider a Riemannian symmetric space G/K of non-compact type. *Assume that G/K is of rank one*. Any zonal spherical function on G/K is expressed as $\varphi_\nu(gK) = \int_K e^{(\nu-\rho)H(gk)}dk$. By changing variables, we have $\varphi_\nu(gK) = \int_{\bar{N}} e^{(\nu-\rho)H(g\bar{n})} e^{-(\nu+\rho)H(\bar{n})} d\bar{n}$. Let $G=KA_pN$ be an Iwasawa decomposition and put $\mathfrak{a}_p = \mathrm{Lie}A_p$. Let $\mathfrak{a}_p^+$ be its positive Weyl chamber. Since $\mathfrak{a}=\mathfrak{a}_p$ from the assumption that G/K is of rank one, we may take $Y \in \mathfrak{a}_p^+$ which is so chosen as in §1. Define $c_{G/K}(\nu) = \lim_{t\to+\infty} e^{-(\nu-\rho)(tY)} \varphi_\nu(e^{tY}K)$ which is convergent when ν varies an open subset of $\mathfrak{a}_{p,c}^*$, has an integral formula $c_{G/K}(\nu) = \int_{\bar{N}} e^{-(\nu+\rho)H(\bar{n})} d\bar{n}$ and is called *Harish-Chandra's c-function*. The quantity $c_{G/K}(\nu)$ plays an important role in the determination of the *Plancherel Measure of G/K* (cf.[HC1]).

I am going to explain two methods of computing $c(\nu)$, the first one is due to Harish-Chandra (HC-method) ([HC1]) and the second one is due to Gindikin-Karpelevic ([GK1]), Helgason ([H1]) and Schiffmann ([Sc1]) (GKHS-method). <u>HC-*Method*</u>: It is known that φ_ν is an eigenfunction of the Laplace- Beltrami operator L of G/K, that is, (F) $L\varphi_\nu = \Delta(L)(\nu)\varphi_\nu$ for a constant $\Delta(L)(\nu)$ depending on ν. If $f_\nu(t) = \varphi_\nu(e^{tY}K)$, then $f_\nu(t)$ satisfies an ordinary differential equation obtained from (F). Taking $x=-(sh t)^2$ as a new variable, we find that the differential equation in question turns out to be a *Gaussian hypergeometric differential equation of* x. From properties of φ_ν, (G.1) $f_\nu(t)$ is a real analytic solution of t near t=0 and (G.2) $f_\nu(0)=1$. Therefore $f_\nu(t) = F(a,b,c;-(sh t)^2)$ for some constants a, b, c depending on m_1, m_2. Then, by using a well-known formula for hypergeometric functions

$$\text{(H)}\qquad F(\alpha,\beta,\gamma;x) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}|x|^{-\alpha}F(\alpha,\alpha-\gamma+1,\alpha-\beta+1;\frac{1}{x}) + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}|x|^{-\beta}F(\beta,\beta-\gamma+1,\beta-\alpha+1;\frac{1}{x})$$

which holds when $x<-1$, we can obtain a concrete formula for $c_{G/K}(\nu)$.

<u>GKHS-*Method*</u>: On the other hand, Gindikin and Karpelevic computed $c(\nu)$ in an alternative way. Their method is available for general Riemannian symmetric space case and is improved by Helgason and Schiffmann in order to study analytic continuations of intertwining operators between principal series representations. In particular, in the rank one case, Helgason and Schiffman showed that the integral $\int_{\bar{N}} e^{-(\nu+\rho)H(\bar{n})}d\bar{n}$ is reduced to $\int_{\mathfrak{g}_{-1}\times\mathfrak{g}_{-2}}\{(1+|X|^2)^2+|T|^2\}^s dXdT$, where $|X|^2$ and $|T|^2$ are positive definite quadratic forms on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$, respectively and that the last integral becomes $\int_0^\infty t_1^{m_1-1}dt_1\int_0^\infty t_2^{m_2-1}\{(1+t_1^2)^2+t_2^2\}^s dt_2$ which is easily computed. As a result, $c_{G/K}(\nu)$ is expressed as a product of Gamma functions.

§3. c-functions for semisimple symmetric spaces. In the sequel, I focus my attention to explain how to obtain concrete forms of c-functions for arbitrary symmetric spaces of split rank one. As explained in §2, Harish-Chandra's c-function is defined as a leading term of the "*asymptotic expansion*" of a zonal spherical function on G/K. Noting this, T. Oshima introduced c-functions for arbitrary semisimple symmetric spaces as "*boundary values*" of certain joint eigenfunctions of all the invariant differential operators. Hereafter, G/H ($H=G^{\sigma}$) is assumed to be simply connected. Let $\mathbb{D}(G/H)$ be the algebra of invariant differential operators on G/H. Since $\mathbb{D}(G/H)$ is commutative, for any algebra homomorphism χ of $\mathbb{D}(G/H)$ to $\mathbb{C}$, it is possible to define a system of differential equations on $G/H: (J_{\chi}) Du=\chi(D)u$ for $^{\forall}D\in\mathbb{D}(G/H)$. If $\mathfrak{j}$ is a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}$, then $\mathrm{Hom}_{\mathbb{C}}(\mathbb{D}(G/H),\mathbb{C})$ is parametrized by $\mathfrak{j}_{\mathbb{C}}^{*}$ due to Harish-Chandra. Let $\mathcal{B}(G/H;(J_{\chi}))$ be the *hyperfunction* solution space to the system (J_{χ}). Then $\mathcal{B}(G/H;(J_{\chi}))$ is a G-space. Now assume that G is linear, namely, $\#F(\mathfrak{g},\mathfrak{h})<\infty$, for simplicity. Due to *Helgason Type Theorem* of Oshima (cf.[O1]) , if χ is "generic", there is a non-zero G-invariant subspace $\mathcal{B}(G/H,\mathcal{O};(J_{\chi}))$ of $\mathcal{B}(G/H;(J_{\chi}))$ for each open H-orbit $\mathcal{O}$ of $G/P_{\sigma,0}$ via *Poisson transformation* and $\mathcal{B}(G/H;(J_{\chi}))$ is a direct sum of $\mathcal{B}(G/H,\mathcal{O};(J_{\chi}))$, where $\mathcal{O}$ runs through all the open H-orbits. Then, basically, to each open H-orbit $\mathcal{O}$, there associates a "*c-function*" *for G/H* which is defined by use of "*boundary value maps*" to a compact boundary component of a compactification of G/H. In the case of $F(\mathfrak{g},\mathfrak{h})=\mathbb{Z}$, a modification is needed in the definition of c-functions. *Note*: As mentioned at the end of §1, there is a unique open H-orbit of $G/P_{\sigma,0}$, whereas sometimes not.

Before treating the general case, I discuss two simpler cases. First consider the case where $(\mathfrak{g},\mathfrak{h})$ is of $\mathfrak{l}_\varepsilon$-type but not Riemannian. Then $(\mathfrak{g},\mathfrak{h})$ is one of $I_i(0,q)$ $(i=1, 2, 3)$ and I_4^2. In this case, GKHS-method is available for the computation of c-functions for G/G^σ because we can obtain an integral formula similar to that for Riemannian symmetric space and the integral reduces to $\int_{\mathfrak{g}_{-1}}(1-|X|^2)_\pm^s dX=\int_0^\infty t^{m_1-1}(1-t^2)_\pm^s dt$ if $(\mathfrak{g},\mathfrak{h})=I_1(0,q)$ and $\int_{\mathfrak{g}_{-1}\times\mathfrak{g}_{-2}}\{(1-|X|^2)^2+|T|^2\}^s dXdT=\int_0^\infty t_1^{m_1-1}dt_1\int_0^\infty t_2^{m_2-1}\{(1-t_1^2)^2+t_2^2\}^s dt_2$ otherwise. In spite that the integrals $\int_0^\infty t^{m_1-1}(1-t^2)_\pm^s dt$, $\int_0^\infty t_1^{m_1-1}dt_1\int_0^\infty t_2^{m_2-1}\{(1-t_1^2)^2+t_2^2\}^s dt_2$ are divergent and lose their meaning in general, it possible to regularlize these divergent integrals by careful investigation of intertwining integrals.

Next consider the case where $(\mathfrak{g},\mathfrak{h})$ is one of $I_i(p,q)$ $(i=1,2,3)$ and the corresponding symmetric space is G/H, where $G=\mathcal{I}nt\,\mathfrak{g}$ and $H=(\mathcal{I}nt\,\mathfrak{g})^\sigma$. In this case, a c-function for G/H is nothing but the leading term of a left K-invariant eigenfunction of the Laplace-Beltrami operator $L_{G/H}$ on G/H. So let $\varphi_s(gH)$ be a left K-invariant function on G/H with the condition (F') $L\varphi_s=\Delta(L)(s)\varphi_s$ for a constant $s\in\mathbb{C}$. Then $\psi_s(gK)=\varphi_s(g^{-1}H)$ is regarded as a left H-invariant eigenfunction of $L_{G/K}$ on the Riemannian symmetric space G/K. From this, it follows that φ_s is real analytic. Because of the decomposition G=KAH, $f_s(t)=\varphi_s(e^{tY}H)$ determines φ_s itself and $f_s(t)$ satisfies a differential equation obtained from (F') which turns out to be a hypergeometric differential equation for the variable $x=-(sh\,t)^2$. So far, the argument is parallel to HC-method for

Riemanniann case except that the value $f_s(0)=\varphi_s(eH)$ is unknown on the contrary to (G.2). But using an integral formula for φ_s and using the c-function for G/K, we can determine the value $f_s(0)$ and therefore the c-function for the space in question.

Return to the general case. For a moment, we assume that *there is a non-zero left K-fixed solution* u_χ *to* (J_χ). If $L_{G/H}$ is the Laplace-Beltrami operator on G/H, then

(K) $\{(\frac{d}{dt})^2+\left(m_1^+ cth t+m_1^- th t+2m_2^+ cth 2t+2m_2^- th 2t\right)\frac{d}{dt}\}f_s(t)=(s^2-\left(\frac{1}{2}m_1+m_2\right)^2)f_s(t)$

where $f_s(t)=u_\chi(e^{tY}H)$ and s is a complex number depending on χ. First consider the case where $m_2^-=0$. Then, rewriting (K) by use of the variable $x=-(sh t)^2$, we have

(K') $[x(1-x)\left(\frac{d}{dx}\right)^2+\{c-(a+b+1)x\}\frac{d}{dx}-ab]v(x)=0,$

where $v(x)=f_s(t)$ and $a=\frac{1}{2}(s+\frac{1}{2}m_1+m_2^+)$, $b=\frac{1}{2}(-s+\frac{1}{2}m_1+m_2^+)$, $c=\frac{1}{2}(m_1^++m_2^++1)$. For the same reason as in the case $I_i(p,q)$ explained before, $f_s(t)$ is a real analytic solution of (K) near t=0 and (K') is a hypergeometric differential equation. Hence we find that $f_s(t)$ coincides with $F(a,b,c;-(sh t)^2)$ up to a constant factor. On the other hand, since $u_\chi(gH)$ is a left K-invariant solution of (F_χ), $u_\chi(gH)$ has an integral representation $u_\chi(gH)=\int_K \xi_s(g^{-1}k)dk$, where ξ_s is a left H-invariant *hyperfunction* section of a certain line bundle over G/P_σ. As in the case of zonal spherical functions on Riemannian symmetric spaces, the last integral is rewritten as $\int_{\bar{N}}\xi_s(g^{-1}\bar{n})e^{-(\nu+\rho)H(\bar{n})}d\bar{n}$, where $\nu\in j_{\mathbb{C}}^*$ depending on s linearly. (*Caution: Don't confuse the group H with the function H(g) for Iwasawa decomposition.*) Then, since $f_s(t)=\int_{\bar{N}}\xi_s(e^{-tY}\bar{n})e^{-(\nu+\rho)H(\bar{n})}d\bar{n}$, it follows that $\lim_{t\to+\infty} e^{-(s+\frac{1}{2}m_1+m_2)t} f(t) =$

$\int_{\bar{N}} e^{-(\nu+\rho)H(\bar{n})}d\bar{n}$. The last integral is nothing but the Harish-Chandra's c-function. On the other hand, $\lim_{t\to+\infty} e^{(s-\frac{1}{2}m_1-m_2)t} f(t)$ is the quantity what we want to compute if the limit exists. With the help of (H), we finally obtain a concrete formula for the leading term of the asymptotics of $u_\chi(gH)$. Next consider the case $m_2^-\neq 0$. In this case, due to (C.2), we find that

$$\begin{aligned}&(\frac{d}{dt})^2+\left(m_1^+ cth t+m_1^- th t+2m_2^+ cth 2t+2m_2^- th 2t\right)\frac{d}{dt}\\&=4\left\{(\frac{d}{dt'})^2+\left((m_1^+ +m_2^+) cth t'+m_2^- th t'\right)\frac{d}{dt'}\right\},\end{aligned}$$

where $t'=2t$. Hence an argument parallel to the case $m_2^-=0$ goes well by changing m_1^+, m_1^-, m_2^+ with $m_1^+ +m_2^+$, m_2^-, 0, respectively.

We give here some remarks on the asymptotics of left K-invariant solutions of (J_χ). (L.1) The relation (C.2) first observed by Oshima is easy to show but plays a crucial role in the determination the leading term in question. It is left open whether for arbitrary parameters $m_1^\pm$, $m_2^\pm$ (that is, forgetting (C.2)), it is possible to obtain a connection formula for solutions of (J) similar to (H) or not. (L.2) It is also important to construct left H-invariant hyperfunction sections $\xi_s(g)$ of certain line bundles over $G/P_{\sigma,0}$. The function $e^{(\lambda-\rho)H(g)}$ is a left K-invariant section and is constructed by using an Iwasawa decomposition. Instead, to construct ξ_s, we need a theorem of J. Bernstein on the analytic continuation of complex powers of polynomials. (L.3) Consider the case SU(2,1)/SO(2,1). In this case, the concrete form of $e^{(s+2)t}u(e^{tY}H)$ is

$$(M_{s,\ell})\ \int_{\mathbb{R}\times\mathbb{C}} \frac{(1+2z^2+(|z|^2-ix)^2)^{(s+\ell)/2}(1+2\bar{z}^2+(|z|^2+ix)^2)^{(s-\ell)/2}}{(1+e^{-t}(|z|^2-ix))^{s+\ell+2}(1+e^{-t}(|z|^2+ix))^{s-\ell+2}}dxdzd\bar{z}$$

<u>substituted with $\ell=0$</u> and and $\lim_{t\to+\infty} e^{(s+2)t}u(e^{tY}H)$ is convergent when

$Re s<-1$ and its limit value is

$$(N_{s,\ell})\ \int_{\mathbb{R}\times\mathbb{C}}(1+2z^2+(|z|^2-ix)^2)^{-s-\ell-2}(1+2\bar{z}^2+(|z|^2+ix)^2)^{-s+\ell-2}dxdzd\bar{z}$$

substituted with $\ell=0$ <u>if it were convergent</u>. It is possible to regularize this definite integral but it seems hard to compute it directly unlike the integrals for Riemannian and $\mathfrak{l}_\varepsilon$-type cases mentioned before. In spite this, due the the armument above based on HC-method, we can compute the value of $(N_{s,0})$ and the result is $2^{-2s}\Gamma(2s+3)\Gamma(s+2)^{-2}$ up to a constant factor independent of s.

Return to the differential equation (J_χ). In the above, we assumed the existence of a non-zero left K-invariant solution to (J_χ). But this does not hold in general. Next I mention two cases for which HC-method or GKHS-method is available.

Assume that $F(\mathfrak{g},\mathfrak{h})=\mathbb{Z}$ *and G/H is simply connected*. Then, due to (A.1), G/H is the universal covering space of one of $SO_0(p+1,2)/SO_0(p+1,1)$, $SU(p+1,1)/SO(p+1,1)$, $SU(p+1,1)\times SU(p+1,1)/\Delta SU(p+1,1)$, $SU(2p+2,2)/Sp(p+1,1)$. In this case, G has infinite center. Let K be the analytic subgorup of G corresponding to $\mathfrak{k}$. Assume that *there is a relative K-invariant solution to the system* (J_χ). Under this assumption, the argument explained before goes well and a similar conclusion is obtained. For example, treat the case G/H=the universal covering space of $SU(2,1)/SO(2,1)$. Then the c-function for G/H is the integral $(N_{s,\ell})$ for arbitrary parameters s, ℓ. A regularization of $(N_{s,\ell})$ equals to

$$\frac{\sin\frac{\pi}{2}(s+\ell)\sin\frac{\pi}{2}(s-\ell)}{\sin\pi s}\ \frac{\Gamma(\frac{s+\ell+2}{2})\Gamma(\frac{s-\ell+2}{2})\Gamma(s+\frac{3}{2})}{\Gamma(\frac{s+\ell+3}{2})\Gamma(\frac{s-\ell+3}{2})\Gamma(s+2)}$$ up to a constant factor.

Next consider the case where $\#F(\mathfrak{g},\mathfrak{h})<\infty$ and $G/P_{\sigma,0}$ has plural open H-orbits. Then G/H is one of $SO_0(p+1,q+1)/SO_0(p+1,q)$ $(q>1)$,

$SL(p+2,\mathbb{R})/GL_+(p+1,\mathbb{R})$, $Sp(p+2,\mathbb{R})/Sp(p+1,\mathbb{R})\times Sp(1,\mathbb{R})$, $F_{4(4)}/Spin_0(5,4)$. At first, note that G/H is of rank one. Now we treat the case $G/H= SO_0(p+1,q+1)/SO_0(p+1,q)$ $(q>1)$. For a generic χ, $\mathcal{B}(G/H;(J_\chi))$ has two G-invariant subspaces. One is spanned by left translations of a K-invariant function but the other is not. Since G/H is of rank one, GKHS-method is available for the determination of the c-function for such a G-invariant subspace. In fact, considering the intertwining integral between degenerate principal series for G, we find that the c-function coincides with the special value $(\tau=0)$ of the definite integral

$$\int_{\mathbb{R}^p\times\mathbb{R}^q}(1+\|x\|^2-\|y\|^2)_\pm^{-s-\tau-(p+q)/2}\{(1+\|x\|^2-\|y\|^2)^2+4\|y\|^2\}^\tau dxdy$$
$$= c\int_0^\infty r_1^{p-1}dr_1\int_0^\infty r_2^{q-1}(1+r_1^2-r_2^2)_\pm^{-s-\tau-(p+q)/2}\{(1+r_1^2-r_2^2)^2+4r_2^2\}^\tau dr_2$$

which is divergent, but is regularizable. Hence we can obtain a concrete form of the c-function in this case. Similar arguments go well for the remaining symmetric spaces $SL(p+2,\mathbb{R})/GL_+(p+1,\mathbb{R})$, $Sp(p+2,\mathbb{R})/Sp(p+1,\mathbb{R})\times Sp(1,\mathbb{R})$, $F_{4(4)}/Spin_0(5,4)$.

Last, I give a comment on general c-functions. For simplicity, assume that $\#F(\mathfrak{g},\mathfrak{h})<\infty$ and that there is an open dense H-orbit of $G/P_{\sigma,0}$. If rank of $G/H = r+1$, then the set $\mathbb{X}=\{\chi;\mathcal{B}(G/H;(J_\chi))\neq 0\}$ has one continuous parameter and r discrete parameters. The discrete parameters run through a set isomorphic to $\mathbb{N}^r$. Hence $\mathbb{X}\cong\mathbb{C}\times\mathbb{N}^r$. Then the c-function is written as $c(s,\mu)$ with $s\in\mathbb{C}$, $\mu\in\mathbb{N}^r$. On the other hand, consider the *Riemannian form* G^d/K^d of G/H and its c-function. Then (O) $c(s,\mu)$ satisfies difference equations same as those for $c_{G^d/K^d}(\nu)$. Since $c_{G^d/K^d}(\nu)$ is known, (O) implies that $c(s,\mu)$ can be obtained if one knows $c(s,0)$ which is the c-function for the case where K-invariant solution to (J_χ) exists. In this way, it is possible to obtain $c(s,\chi)$ for all parameters $\chi \in \mathbb{X}$.

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